

# UNIFORM BOUNDS FOR THE HEAT CONTENT OF OPEN SETS IN EUCLIDEAN SPACE

M. VAN DEN BERG \*, K. GITTINS

**ABSTRACT.** We obtain (i) lower and upper bounds for the heat content of an open set in  $\mathbb{R}^m$  with  $R$ -smooth boundary and finite Lebesgue measure, (ii) a necessary and sufficient geometric condition for finiteness of the heat content in  $\mathbb{R}^m$ , and corresponding lower and upper bounds, (iii) lower and upper bounds for the heat loss of an open set in  $\mathbb{R}^m$  with finite Lebesgue measure.

## 1. INTRODUCTION

In this paper we obtain results for the heat content of an open and bounded set  $D$  in Euclidean space  $\mathbb{R}^m$ ,  $m = 2, 3, \dots$ , where  $D$  has initial temperature 1 and the complement of  $D$  has initial temperature 0. We denote the fundamental solution of the heat equation on  $\mathbb{R}^m$  by

$$p(x, y; t) = (4\pi t)^{-m/2} e^{-|x-y|^2/(4t)},$$

and define

$$(1) \quad u_D(x; t) = \int_D dy p(x, y; t).$$

It is standard to check (see Chapter 2 in [12]) that

$$(2) \quad \Delta u_D = \frac{\partial u_D}{\partial t}, \quad x \in \mathbb{R}^m, \quad t > 0,$$

and that

$$(3) \quad \lim_{t \downarrow 0} u_D(x; t) = \mathbf{1}_D(x), \quad x \in \mathbb{R}^m - \partial D,$$

where  $\partial D$  is the boundary of  $D$ .

We define the *heat content of  $D$  in  $\mathbb{R}^m$  at  $t$*  by

$$(4) \quad H_D(t) = \int_D dx u_D(x; t).$$

Thus, if  $u_D(x; t)$  represents the temperature at point  $x \in \mathbb{R}^m$  at time  $t$  with initial condition (3), then the heat content of  $D$  in  $\mathbb{R}^m$  at time  $t$  represents the amount of heat in  $D$  at time  $t$ . By (1) and (4) we see that

$$(5) \quad H_D(t) = \int_D dx \int_D dy p(x, y; t).$$

By the heat semigroup property we have that

$$(6) \quad p(x, y; t) = \int_{\mathbb{R}^m} dz p(x, z; t/2) p(z, y; t/2).$$

By Tonelli's Theorem and (1), (5) and (6) we conclude that

$$H_D(t) = \|u_D(\cdot; t/2)\|_{L^2(\mathbb{R}^m)}^2.$$

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*Date:* 19 February 2015.

*2010 Mathematics Subject Classification.* Primary: 35K05; Secondary: 35K20.

*Key words and phrases.* Heat content, Euclidean space.

\* Partially supported by The Leverhulme Trust, International Network Grant *Laplacians, Random Walks, Bose Gas, Quantum Spin Systems*.

Preunkert [15] defines the  $L^2$ -curve of the set  $D$  as the map  $t \mapsto \|u_D(\cdot; t/2)\|_{L^2(\mathbb{R}^m)}$ . The results for the  $L^2$ -curve in Theorem 2.4 of [16] imply that if  $D$  is an open, bounded subset of  $\mathbb{R}^m$  with  $C^{1,1}$ -boundary  $\partial D$  then

$$(7) \quad H_D(t) = |D| - \pi^{-1/2} \mathcal{P}(D) t^{1/2} + o(t^{1/2}), \quad t \downarrow 0,$$

where  $|D|$  denotes the Lebesgue measure of  $D$  and  $\mathcal{P}(D)$  denotes the perimeter of  $D$ . Note that since  $\partial D$  is Lipschitz,  $\mathcal{P}(D) = \mathcal{H}^{m-1}(\partial D)$ , the  $(m-1)$ -dimensional Hausdorff measure of the boundary (see Remark (ii) on p.183 in [13]).

Initial value problems of the type (2)-(3) have been studied in the much wider context of operators of Laplace type on compact Riemannian manifolds [8]. The results of that paper imply that if  $D$  is open, bounded in  $\mathbb{R}^m$  with  $C^\infty$  boundary then there exist geometric invariant  $h_0, h_1, \dots$  such that for any  $J \in \mathbb{N}$ ,

$$(8) \quad H_D(t) = \sum_{j=0}^J h_j t^{j/2} + O(t^{(J+1)/2}), \quad t \downarrow 0.$$

Furthermore if  $\partial D$  is oriented by a unit inward pointing normal vector field and if  $\{k_1(s), \dots, k_{m-1}(s)\}$  are the principal curvatures at  $s \in \partial D$  then  $h_0 = |D|$ ,  $h_1 = -\pi^{-1/2} \int_{\partial D} d\mathcal{H}^{m-1}(s)$ ,  $h_2 = 0$  and

$$h_3 = - \int_{\partial D} d\mathcal{H}^{m-1}(s) \left( \frac{5}{32} \left( \sum_{i=1}^{m-1} k_i(s) \right)^2 + \frac{1}{16} \sum_{i=1}^{m-1} k_i^2(s) \right).$$

For further results in the Riemannian manifold setting we refer to [11] and [9].

We note that  $p(x, y; t)$  is also the transition density of Brownian motion  $(B(t), t \geq 0, \mathbb{P}_x, x \in \mathbb{R}^m)$  associated to the Laplacian. In fact  $u_D(x; t) = \mathbb{P}_x(B(t) \in D)$ , and  $|D|^{-1} H_D(t)$  is precisely this probability averaged over all starting points with uniform density.

The knowledge of the asymptotic behaviour of  $H_D(t)$ ,  $t \downarrow 0$  as in (7) or (8) does not give any information about its actual numerical value. Furthermore, due to the discontinuity of  $\mathbb{1}_D$ , the implementation of numerical schemes is non-trivial in the small  $t$  regime.

In this paper, we address this issue and obtain uniform bounds for  $H_D(t)$  under the hypotheses that either  $D$  has  $R$ -smooth boundary  $\partial D$  and finite Lebesgue measure  $|D|$ , or  $D$  is an arbitrary open set satisfying a geometrical integrability condition. These bounds are uniform in both the geometrical data of  $D$  and  $t$ , and the  $m$ -dependent constants are explicit.

**Definition 1.** *An open set  $D \subset \mathbb{R}^m$ ,  $m \geq 2$ , has  $R$ -smooth boundary if at any point  $x_0 \in \partial D$ , there are two open balls  $B_1, B_2$  with radii  $R$  such that  $B_1 \subset D$ ,  $B_2 \subset \mathbb{R}^m - \bar{D}$  and  $\bar{B}_1 \cap \bar{B}_2 = \{x_0\}$ .*

**Theorem 2.** *Let  $D \subset \mathbb{R}^m$ ,  $m \geq 2$ , be an open set with  $R$ -smooth boundary  $\partial D$  and finite Lebesgue measure  $|D|$ . Then for all  $t > 0$ ,*

$$\left| H_D(t) - |D| + \pi^{-1/2} \mathcal{H}^{m-1}(\partial D) t^{1/2} \right| \leq m^3 2^{m+2} |D| R^{-2} t.$$

Bounds of this type have been obtained in [2] for the trace of the Dirichlet heat semigroup and in [6] for the heat content with Dirichlet boundary conditions respectively.

Examples of open sets with infinite volume but with finite heat content in  $\mathbb{R}^m$  for any  $t > 0$  have been given in Theorem 4 of [5]. The mechanism for that phenomenon is very different from the one where one imposes Dirichlet cooling conditions on the boundary. Efficient cooling takes place if the boundary is not too thin. The latter condition can be phrased in terms of a capacity density condition for the boundary or a strong Hardy condition for the quadratic form associated to the Dirichlet Laplacian. For further details we refer to [3, 4, 7] and [10]. In the setting of (2), (3) and (4) above, the mechanism for efficient heat loss is that the complement is not too small. With this in mind, we introduce the following.

**Definition 3.** For an open set  $D \subset \mathbb{R}^m$ ,  $R > 0$  and  $x \in D$  we define

$$\mu_D(x; R) = |B(x; R) \cap D|,$$

where  $B(x; R) = \{y \in \mathbb{R}^m : |x - y| < R\}$ .

Our second result is the following.

**Theorem 4.** Let  $D$  be an open set in  $\mathbb{R}^m$ , and let  $t > 0$ . Then

(i)  $H_D(t) < \infty$  if and only if  $\int_D dx \mu_D(x; (8mt)^{1/2}) < \infty$ . If  $\int_D dx \mu_D(x; (8mt)^{1/2}) < \infty$  then there exist constants  $c_1(m)$  and  $c_2(m)$  such that

$$(9) \quad c_1(m)t^{-m/2} \int_D dx \mu_D(x; (8mt)^{1/2}) \leq H_D(t) \leq c_2(m)t^{-m/2} \int_D dx \mu_D(x; (8mt)^{1/2}),$$

where

$$c_1(m) = e^{-2m} (4\pi)^{-m/2},$$

and

$$c_2(m) = \left(1 - 2^m e^{-3m/2}\right)^{-1} (4\pi)^{-m/2}.$$

(ii) If  $0 < t_2 \leq t_1$  and if  $H_D(t_1) < \infty$  then

$$(10) \quad H_D(t_2) \leq \frac{c_2(m)}{c_1(m)} \left(\frac{t_1}{t_2}\right)^{m/2} H_D(t_1).$$

Since  $t \mapsto H_D(t)$  is monotonically decreasing (see [15]), (ii) implies that if  $H_D(t)$  is finite for some  $t > 0$  then it is finite for all  $t > 0$ . This is in contrast with the situation where the boundary is kept at fixed temperature 0. See for example Theorem 5.5 in [6].

While Theorem 4 holds in the case where  $D$  has finite Lebesgue measure, we have the trivial upper bound

$$(11) \quad H_D(t) \leq |D|$$

in that case, which is sharper than the upper bound in (9). Similarly we have by Proposition 9(i) in [5] that

$$(12) \quad H_D(t) \geq |D| - 2^{m/2} \int_D dx e^{-\delta(x)^2/(8t)},$$

where

$$\delta(x) = \min\{|x - y| : y \in \mathbb{R}^m - D\}$$

. The latter implies  $\liminf_{t \downarrow 0} H_D(t) = |D|$ , which is sharper than the lower bound in (9). So Theorem 4 is of interest only in the case where  $D$  has infinite Lebesgue measure.

The choice  $R = (8mt)^{1/2}$  in the application of Definition 3 in Theorem 4 is natural since diffusion or transport of heat for small  $t$  takes place on a timescale  $t^{1/2}$ . However, the constant  $(8m)^{1/2}$  is somewhat arbitrary: any sufficiently large numerical constant would suffice. Of course the choice of constant affects the numerical values of  $c_1(m)$  and  $c_2(m)$  in Theorem 4.

We recall the examples in Theorem 4 of [5] with finite heat content in  $\mathbb{R}^m$  and with infinite volume. Let  $\Sigma$  be an open, bounded convex set in  $\mathbb{R}^{m-1}$  and let

$$(13) \quad \Omega(\alpha, \Sigma) = \{(x, x') \in \mathbb{R}^m : x > 1, x' \in x^{-\alpha} \Sigma\},$$

where  $\alpha$  is a fixed positive constant. Then  $|\Omega(\alpha, \Sigma)| = +\infty$  if and only if  $\alpha < (m-1)^{-1}$ . We have the following.

**Proposition 5.** Let  $\Omega(\alpha, \Sigma)$  as in (13) and let  $t > 0$ . Then

$$(14) \quad H_{\Omega(\alpha, \Sigma)}(t) < \infty \Leftrightarrow \alpha > (2(m-1))^{-1}.$$

If  $(2(m-1))^{-1} < \alpha < (m-1)^{-1}$  then

$$(15) \quad H_{\Omega(\alpha, \Sigma)}(t) \asymp t^{((m-1)\alpha-1)/(2\alpha)}, \quad t \downarrow 0.$$

The precise asymptotic behaviour of  $H_{\Omega(\alpha, \Sigma)}(t)$  was obtained in Theorem 4 (ii) of [5]. However, the range of  $\alpha$  for which we shall prove (14) and (15) was incorrectly stated in parts (i) and (ii) of that theorem.

If  $D$  has finite Lebesgue measure then it is convenient to define the *heat loss of  $D$  in  $\mathbb{R}^m$  at  $t$*  by

$$(16) \quad F_D(t) = |D| - H_D(t).$$

We see that (11) and (12) imply that

$$0 \leq F_D(t) \leq 2^{m/2} \int_D dx e^{-\delta(x)^2/(8t)}.$$

Moreover,  $t \mapsto F_D(t)$  is strictly increasing. In Theorem 7 below we identify the quantity which controls the heat loss of  $D$  in  $\mathbb{R}^m$  at  $t$ , and which motivates the following.

**Definition 6.** For an open set  $D \subset \mathbb{R}^m$  with finite Lebesgue measure,  $R > 0$ , and  $x \in D$  we define

$$\nu_D(x; R) = |B(x; R) \cap (\mathbb{R}^m - D)|.$$

**Theorem 7.** If  $D$  is an open set in  $\mathbb{R}^m$  with finite Lebesgue measure then there exist constants  $d_1(m)$  and  $d_2(m)$  such that

$$(17) \quad d_1(m)t^{-m/2} \int_D dx \nu_D(x; 4(mt)^{1/2}) \leq F_D(t) \leq d_2(m)t^{-m/2} \int_D dx \nu_D(x; 4(mt)^{1/2}),$$

where

$$d_1(m) = e^{-4m} (4\pi)^{-m/2},$$

and

$$d_2(m) = \left(1 - 2^{(m+2)/2} e^{-2m}\right)^{-1} (4\pi)^{-m/2}.$$

We conclude this section with the following.

**Proposition 8.** Let  $D$  be an open set in  $\mathbb{R}^m$  with finite Lebesgue measure, and let  $1 \leq p < \infty$ . Then (3) implies  $\lim_{t \downarrow 0} \|u_D(\cdot; t) - \mathbf{1}_D(\cdot)\|_{L^p(\mathbb{R}^m)} = 0$ .

*Proof.* We note that  $0 < u_D(x; t) \leq 1$ . Hence  $|u_D(x; t) - 1| \leq 1$ , and

$$\begin{aligned} \|u_D(\cdot; t) - \mathbf{1}_D(\cdot)\|_{L^p(\mathbb{R}^m)}^p &= \int_D dx |u_D(x; t) - 1|^p + \int_{\mathbb{R}^m - D} dx u_D(x; t)^p \\ &\leq \int_D dx |u_D(x; t) - 1| + \int_{\mathbb{R}^m - D} dx u_D(x; t) \\ &= \int_D dx |u_D(x; t) - 1| + \int_{\mathbb{R}^m} dx u_D(x; t) - \int_D dx u_D(x; t) \\ &= 2 \int_D dx |u_D(x; t) - 1|. \end{aligned}$$

The assertion follows by Lebesgue's Dominated Convergence Theorem and (3).  $\square$

This paper is organised as follows. In Section 2 we state and prove some geometric facts for open sets with  $R$ -smooth boundary and finite Lebesgue measure (Proposition 9). We defer the proof of Theorem 2 to Section 3, where we give lower and upper bounds for  $u_D(x; t)$  (Lemma 10, Lemma 11). We then integrate these bounds with respect to  $x$  to obtain lower and upper bounds for  $H_D(t)$  (Lemma 12, Lemma 13) which imply Theorem 2 as advertised. We defer the proofs of Theorem 4 and of Theorem 7 to Section 4 respectively. There we will also prove Proposition 5.

## 2. GEOMETRICAL RESULTS FOR OPEN SETS WITH $R$ -SMOOTH BOUNDARY AND FINITE LEBESGUE MEASURE

In what follows,  $\omega_m$  denotes the volume of a unit ball  $B(x; 1)$  in  $\mathbb{R}^m$  and we define

$$\text{diam}(D) = \sup\{|x - y| : x \in D, y \in D\},$$

$$D_r = \{x \in D : \delta(x) > r\}$$

and, for  $\epsilon > 0$ ,

$$D(\epsilon) = \cap_{\{r : r < \epsilon\}} D_r = \{x \in D : \delta(x) \geq \epsilon\}.$$

**Proposition 9.** *If  $D \subset \mathbb{R}^m$ ,  $m \geq 2$  is an open set with  $R$ -smooth boundary  $\partial D$  and finite Lebesgue measure  $|D|$  then:*

- (i)  *$D$  has at most  $\omega_m^{-1} R^{-m} |D|$  components.*
- (ii) *If  $D^i$  is a component of  $D$  then*

$$(18) \quad \text{diam}(D^i) \leq \frac{|D^i| + (2\omega_{m-1} - \omega_m)R^m}{\omega_{m-1}R^{m-1}},$$

*with equality if  $D^i$  is an  $R$ -neighbourhood of a straight line segment.*

- (iii)  *$D$  is a  $C^{1,1}$  domain.*
- (iv)  *$\mathcal{H}^{m-1}(\partial D) = \mathcal{P}(D)$ .*
- (v) *For  $0 < r < R$ ,*

$$(19) \quad \mathcal{H}^{m-1}(\partial D^i) \left( \frac{R-r}{R} \right)^{m-1} \leq \mathcal{H}^{m-1}(\partial(D_r^i)) \leq \mathcal{H}^{m-1}(\partial D^i) \left( \frac{R}{R-r} \right)^{m-1},$$

*and*

$$(20) \quad \mathcal{H}^{m-1}(\partial D) \leq \frac{m|D|}{R}.$$

*Proof.* (i) Let  $D = \cup_{i \in I} D^i$  where  $i \in I$  and  $I$  is some indexing set. Since the components of  $D$  are open,  $I$  is at most countable. Let  $n(I)$  denote the number of elements in  $I$ , i.e. the number of components of  $D$ . By the  $R$ -smoothness of  $\partial D$ , we have that for each  $x \in \partial D$ , there is a ball  $B_1$  of radius  $R$  such that  $x \in \bar{B}_1$  and  $B_1 \subset D^i$  for some  $i \in I$ . From this we deduce that  $|D^i| \geq |B_1| = \omega_m R^m$ . So

$$|D| = \sum_{i \in I} |D^i| \geq \sum_{i \in I} \omega_m R^m = n(I) \omega_m R^m$$

which implies the required estimate.

(ii) We now consider an arbitrary component and denote it by  $D$ . Since  $D$  is open and connected in  $\mathbb{R}^m$ ,  $D$  is path connected. Let  $x_0 \in D(R)$  be arbitrary. If  $D$  is not bounded then there exists a path  $\gamma : [0, \infty) \rightarrow D$  with  $\gamma(0) = x_0$  and  $\lim_{t \rightarrow \infty} |\gamma(t) - x_0| = +\infty$ . Let  $x_n \in \gamma([0, \infty))$ ,  $n \in \mathbb{N}$  be such that

$$|x_n - x_0| = 4nR.$$

By the triangle inequality

$$(21) \quad |x_n - x_k| \geq 4|n - k|R.$$

If  $B(x_j; R)$  is not contained in  $D$ , then let  $z_j$  be a point of  $\partial D$  such that  $|z_j - x_j| = \delta(x_j) < R$ . By the  $R$ -smoothness of  $\partial D$ , there exists a point  $y_j \in D$  such that  $B(y_j; R) \subset D$  and  $|z_j - y_j| = R$ . Then  $|y_j - x_j| < R$ . If  $B(x_j; R)$  is contained in  $D$  then we put  $y_j = x_j$ . This defines a sequence of points  $(y_j)$  in  $D$  such that  $|y_j - x_j| < R$ . By (21) and the triangle inequality we have that, for  $n \neq k$

$$|y_n - y_k| \geq ||x_n - x_k| - |y_n - x_n| - |y_k - x_k|| \geq |4|n - k|R - 2R| \geq 2|n - k|R.$$

We conclude that open balls  $\{B(y_n; R), n \in \mathbb{N}\}$  are pairwise disjoint and contained in  $D$ . Hence for any  $N \in \mathbb{N}$ ,  $D \supset \cup_{n=1}^N B(y_n; R)$ . So  $|D| \geq N\omega_m R^m$ , which is a contradiction for  $N$  sufficiently large. Therefore  $D$  is bounded.

For any point  $z \in \partial D$ , let  $z_R$  be such that  $|z - z_R| = R$  and  $B(z_R; R) \subset D$ . The map  $z \mapsto z_R$  is continuous. Since  $\partial D$  is path connected, its image under  $z \mapsto z_R$  is path connected. We conclude that both  $\partial D(R)$  and  $D(R)$  are path connected. Let  $p$  and  $q$  be points in  $\partial D$  such that  $\text{diam}(D) = |p - q|$ . Let  $\varphi : [0, \infty) \rightarrow D(R)$  be a path from  $p_R$  to  $q_R$ . Let  $P_\xi$ ,  $0 \leq \xi \leq \text{diam}(D)$ , be the plane perpendicular to the line segment  $[p, q]$  such that it intersects this line segment at a point which has distance  $\xi$  to  $p$ . For  $R \leq \xi \leq \text{diam}(D) - R$ ,  $P_\xi$  intersects  $\varphi$  in a point which belongs to  $D(R)$ . Hence for

$R \leq \xi \leq \text{diam}(D) - R$ ,  $\mathcal{H}^{m-1}(P_\xi \cap D) \geq \omega_{m-1} R^{m-1}$ . We conclude that

$$\begin{aligned}
|D| &= \int_0^{\text{diam}(D)} d\xi \mathcal{H}^{m-1}(P_\xi \cap D) \\
&\geq \frac{1}{2} \omega_m R^m + \int_R^{\text{diam}(D)-R} d\xi \mathcal{H}^{m-1}(P_\xi \cap D) + \frac{1}{2} \omega_m R^m \\
&\geq \omega_m R^m + \int_R^{\text{diam}(D)-R} d\xi \omega_{m-1} R^{m-1} \\
&= \omega_m R^m + \omega_{m-1} (\text{diam}(D) - 2R) R^{m-1}.
\end{aligned}$$

This implies (18), and the assertion of equality for an  $R$ -neighbourhood of a straight line segment.

(iii) We have by (i) and (ii) that  $D$  is bounded. Since  $D$  is  $R$ -smooth we infer by Lemma 2.2 in [1] that  $D$  is a  $C^{1,1}$  domain.

(iv) This follows from the fact that  $D$  is a bounded  $C^{1,1}$  domain and by Remark (ii) on p.183 in [13].

(v) Inequality (19) was proved in Lemma 5 of [2] for a bounded, open set in  $\mathbb{R}^m$  with  $R$ -smooth boundary. As a consequence of (19), we have that (see (6.15) in [6])

$$\mathcal{H}^{m-1}(\partial D^i) \leq \frac{m|D^i|}{R}.$$

Since the components of  $D$  and their respective boundaries are disjoint we have that

$$(22) \quad \mathcal{H}^{m-1}(\partial D) = \sum_i \mathcal{H}^{m-1}(\partial D^i) \leq \sum_i \frac{m|D^i|}{R} = \frac{m|D|}{R}.$$

□

### 3. PROOF OF THEOREM 2

Firstly, we set up a convenient coordinate system for the calculations which follow. Let  $x \in D$  satisfy  $\delta(x) < \frac{R}{2}$ , and  $x_0 \in \partial D$  be the point such that  $\delta(x) = |x - x_0|$ . Choose coordinates  $(\zeta, \hat{\zeta})$  of  $\mathbb{R}^m$  where  $\zeta$  is the direction of the outward pointing normal to  $\partial D$  at  $x_0$ ,  $\hat{\zeta}$  represents the  $m-1$  orthogonal directions to  $\zeta$  and  $x = (0, 0)$ . In these coordinates, the centre of  $B_1$  is  $(-(R - \delta(x)), 0)$  and the centre of  $B_2$  is  $(\delta(x) + R, 0)$ . We suppress the  $x$ -dependence of  $B_1$  and  $B_2$  respectively throughout.

In the following two lemmas we give pointwise lower and upper bounds for  $u_D(x; t)$  respectively when  $\delta(x) < \frac{R}{2}$ .

**Lemma 10.** *Let  $D \subset \mathbb{R}^m$ ,  $m \geq 2$ , be an open set with  $R$ -smooth boundary  $\partial D$  and let  $x \in D$  such that  $\delta(x) < \frac{R}{2}$ . Then*

$$\begin{aligned}
\int_D dy p(x, y; t) &\geq 1 - (4\pi t)^{-1/2} \int_{\delta(x)}^\infty d\zeta e^{-\zeta^2/(4t)} - \frac{\sqrt{2}}{2} e^{-R^2/(8t)} \\
&\quad - (4\pi t)^{-m/2} \int_{-R}^{\delta(x)} d\zeta e^{-\zeta^2/(4t)} \int_{\{|\hat{\zeta}| > ((\delta(x) - \zeta)\frac{R}{2})^{1/2}\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)}.
\end{aligned}$$

*Proof.* First suppose that  $D$  is connected. The set of points  $(\zeta, \hat{\zeta}) \in \mathbb{R}^m$  with fixed  $\zeta \in [\delta(x) - 2R, \delta(x)]$  intersects the ball  $B_1$  in an  $(m-1)$ -dimensional ball of radius  $\eta$ , where  $\eta = (\delta(x) - \zeta)^{1/2} (2R + \zeta - \delta(x))^{1/2}$ . Below we use the fact that for  $\zeta > -R$  and

$\delta(x) < \frac{R}{2}$ ,  $\eta \geq ((\delta(x) - \zeta)\frac{R}{2})^{1/2}$ . So

$$\begin{aligned}
\int_D dy p(x, y; t) &\geq \int_{B_1} dy p(x, y; t) \\
&= (4\pi t)^{-m/2} \int_{\delta(x)-2R}^{\delta(x)} d\zeta e^{-\zeta^2/(4t)} \int_{\{|\hat{\zeta}| < \eta\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)} \\
&\geq (4\pi t)^{-m/2} \int_{-R}^{\delta(x)} d\zeta e^{-\zeta^2/(4t)} \int_{\{|\hat{\zeta}| < ((\delta(x)-\zeta)\frac{R}{2})^{1/2}\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)} \\
&= (4\pi t)^{-1/2} \int_{-R}^{\delta(x)} d\zeta e^{-\zeta^2/(4t)} \\
&\quad - (4\pi t)^{-m/2} \int_{-R}^{\delta(x)} d\zeta e^{-\zeta^2/(4t)} \int_{\{|\hat{\zeta}| > ((\delta(x)-\zeta)\frac{R}{2})^{1/2}\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)} \\
&= 1 - (4\pi t)^{-1/2} \int_{\delta(x)}^{\infty} d\zeta e^{-\zeta^2/(4t)} - (4\pi t)^{-1/2} \int_{-\infty}^{-R} d\zeta e^{-\zeta^2/(4t)} \\
&\quad - (4\pi t)^{-m/2} \int_{-R}^{\delta(x)} d\zeta e^{-\zeta^2/(4t)} \int_{\{|\hat{\zeta}| > ((\delta(x)-\zeta)\frac{R}{2})^{1/2}\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)} \\
&\geq 1 - (4\pi t)^{-1/2} \int_{\delta(x)}^{\infty} d\zeta e^{-\zeta^2/(4t)} - \frac{\sqrt{2}}{2} e^{-R^2/(8t)} \\
&\quad - (4\pi t)^{-m/2} \int_{-R}^{\delta(x)} d\zeta e^{-\zeta^2/(4t)} \int_{\{|\hat{\zeta}| > ((\delta(x)-\zeta)\frac{R}{2})^{1/2}\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)}.
\end{aligned}$$

If  $D$  is not connected, then  $D = \cup_{i=1}^N D^i$  for some  $N \in \mathbb{N}$ . Since  $x \in D$ ,  $x \in D^i$  for some  $i \in \{1, \dots, N\}$  with  $\delta(x) < \frac{R}{2}$ . By the  $R$ -smoothness of the boundary, there is a ball  $B_1^i$  of radius  $R$  such that  $B_1^i \subset D^i$ . So we recover Lemma 10 in that case by the computation above.  $\square$

**Lemma 11.** *Let  $D \subset \mathbb{R}^m$ ,  $m \geq 2$ , be an open set with  $R$ -smooth boundary  $\partial D$ , and let  $x \in D$  such that  $\delta(x) < \frac{R}{2}$ . Then*

$$\begin{aligned}
\int_D dy p(x, y; t) &\leq 1 - (4\pi t)^{-1/2} \int_{\delta(x)}^{\infty} d\zeta e^{-\zeta^2/(4t)} + \frac{\sqrt{2}}{2} e^{-R^2/(8t)} \\
&\quad + (4\pi t)^{-m/2} \int_{\delta(x)}^{\delta(x)+R} d\zeta e^{-\zeta^2/(4t)} \int_{\{|\hat{\zeta}| > ((\zeta-\delta(x))R)^{1/2}\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)}.
\end{aligned}$$

*Proof.* First suppose that  $D$  is connected. Let  $H_x = \{(\zeta, \hat{\zeta}) \mid -\infty < \zeta < \delta(x)\}$ ,  $\tilde{H}_x = \{(\zeta, \hat{\zeta}) \mid \delta(x) + R < \zeta < \infty\}$  and  $S_x$  be the slice of width  $R$  with  $\partial S_x$  parallel to  $\partial H_x$  excluding the ball  $B_2$ . The set of points  $(\zeta, \hat{\zeta}) \in \mathbb{R}^m$  with fixed  $\zeta \in [\delta(x), \delta(x)+R]$  intersects the ball  $B_2$  in an  $(m-1)$ -dimensional ball of radius  $\eta$ , where  $\eta = (\zeta - \delta(x))^{1/2}(2R + \delta(x) -$

$\zeta)^{1/2}$ . Below we use the fact that for  $\zeta < \delta(x) + R$ ,  $\eta \geq ((\zeta - \delta(x))R)^{1/2}$ . So

$$\begin{aligned}
\int_D dy p(x, y; t) &\leq \int_{\mathbb{R}^m - B_2} dy p(x, y; t) \\
&\leq \int_{H_x} dy p(x, y; t) + \int_{\tilde{H}_x} dy p(x, y; t) + \int_{S_x} dy p(x, y; t) \\
&= (4\pi t)^{-1/2} \int_{-\infty}^{\delta(x)} d\zeta e^{-\zeta^2/(4t)} + (4\pi t)^{-1/2} \int_{\delta(x)+R}^{\infty} d\zeta e^{-\zeta^2/(4t)} \\
&\quad + (4\pi t)^{-m/2} \int_{\delta(x)}^{\delta(x)+R} d\zeta e^{-\zeta^2/(4t)} \int_{\{|\hat{\zeta}| > \eta\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)} \\
&\leq 1 - (4\pi t)^{-1/2} \int_{\delta(x)}^{\infty} d\zeta e^{-\zeta^2/(4t)} + \frac{\sqrt{2}}{2} e^{-R^2/(8t)} \\
&\quad + (4\pi t)^{-m/2} \int_{\delta(x)}^{\delta(x)+R} d\zeta e^{-\zeta^2/(4t)} \int_{\{|\hat{\zeta}| > ((\zeta - \delta(x))R)^{1/2}\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)}.
\end{aligned}$$

If  $D$  is not connected, then  $D = \cup_{i=1}^N D^i$  for some  $N \in \mathbb{N}$ . Since  $x \in D$ ,  $x \in D^i$  for some  $i \in \{1, \dots, N\}$  such that  $\delta(x) < \frac{R}{2}$ . By the  $R$ -smoothness of the boundary, there is a ball  $B_2^i$  of radius  $R$  such that  $B_2^i \subset \mathbb{R}^m - \bar{D}^i$  and  $D \subset \mathbb{R}^m - B_2^i$ . So we recover Lemma 11 in that case by the computation above.  $\square$

In the following two lemmas, we give lower and upper bounds for  $H_D(t)$  which imply Theorem 2. The  $R$ -smoothness of the boundary  $\partial D$  ensures that the components  $D^i$  of  $D$  are sufficiently far apart so that the heat flow from one component has a negligible effect on the heat flow of another component. We also use (19), (22) and the additivity properties of the Hausdorff and Lebesgue measures.

**Lemma 12.** *Let  $D \subset \mathbb{R}^m$ ,  $m \geq 2$ , be an open set with  $R$ -smooth boundary  $\partial D$  and finite Lebesgue measure  $|D|$ . Then*

$$\begin{aligned}
H_D(t) &\geq |D| - \pi^{-1/2} \mathcal{H}^{m-1}(\partial D) t^{1/2} - 2^{m/2} |D_{R/2}| e^{-R^2/(32t)} \\
&\quad - (m-1) 2^{m-2} \mathcal{H}^{m-1}(\partial D) R^{-1} t - \frac{\sqrt{2}}{2} |D - D_{R/2}| e^{-R^2/(8t)} \\
&\quad - 4(m-1)(1 + (m-1) 2^{m-2}) \mathcal{H}^{m-1}(\partial D) R^{-1} t.
\end{aligned}$$

*Proof.* First suppose that  $D$  is connected. If  $\delta(x) > \frac{R}{2}$  then by Proposition 9(i) in [5], we have that

$$\int_D dy p(x, y; t) \geq 1 - 2^{m/2} e^{-R^2/(32t)},$$

and so

$$\int_{D_{R/2}} dx \int_D dy p(x, y; t) \geq |D_{R/2}| - 2^{m/2} |D_{R/2}| e^{-R^2/(32t)}.$$



If  $\delta(x) < \frac{R}{2}$ , integrating the result of Lemma 10 with respect to  $x$  over  $D - D_{R/2}$  and using Lemma 6.7 in [6] we obtain that

$$\begin{aligned}
& \int_{D-D_{R/2}} dx \int_D dy p(x, y; t) \\
& \geq |D - D_{R/2}| - (4\pi t)^{-1/2} \int_{D-D_{R/2}} dx \int_{\delta(x)}^\infty d\zeta e^{-\zeta^2/(4t)} - \frac{\sqrt{2}}{2} |D - D_{R/2}| e^{-R^2/(8t)} \\
& \quad - (4\pi t)^{-m/2} \int_{D-D_{R/2}} dx \int_{-R}^{\delta(x)} d\zeta e^{-\zeta^2/(4t)} \int_{\{|\hat{\zeta}| > ((\delta(x)-\zeta)\frac{R}{2})^{1/2}\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)} \\
& = |D - D_{R/2}| - (4\pi t)^{-1/2} \int_0^{R/2} dr \mathcal{H}^{m-1}(\partial D_r) \int_r^\infty d\zeta e^{-\zeta^2/(4t)} \\
& \quad - \frac{\sqrt{2}}{2} |D - D_{R/2}| e^{-R^2/(8t)} \\
& \quad - (4\pi t)^{-m/2} \int_0^{R/2} dr \mathcal{H}^{m-1}(\partial D_r) \int_{-R}^r d\zeta e^{-\zeta^2/(4t)} \int_{\{|\hat{\zeta}| > ((r-\zeta)\frac{R}{2})^{1/2}\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)} \\
& \geq |D - D_{R/2}| - (4\pi t)^{-1/2} \mathcal{H}^{m-1}(\partial D) \int_0^{R/2} dr \left( \frac{R}{R-r} \right)^{m-1} \int_r^\infty d\zeta e^{-\zeta^2/(4t)} \\
& \quad - \frac{\sqrt{2}}{2} |D - D_{R/2}| e^{-R^2/(8t)} - (4\pi t)^{-m/2} \mathcal{H}^{m-1}(\partial D) \\
& \quad \times \int_0^{R/2} dr \left( \frac{R}{R-r} \right)^{m-1} \int_{-R}^r d\zeta e^{-\zeta^2/(4t)} \int_{\{|\hat{\zeta}| > ((r-\zeta)\frac{R}{2})^{1/2}\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)} \\
& \geq |D - D_{R/2}| - (4\pi t)^{-1/2} \mathcal{H}^{m-1}(\partial D) \int_0^{\frac{R}{2}} dr \left( 1 + (m-1)2^{m-1} \frac{r}{R} \right) \int_r^\infty d\zeta e^{-\zeta^2/(4t)} \\
& \quad - \frac{\sqrt{2}}{2} |D - D_{R/2}| e^{-R^2/(8t)} - (4\pi t)^{-m/2} \mathcal{H}^{m-1}(\partial D) \\
& \quad \times \int_0^{R/2} dr \left( 1 + (m-1)2^{m-1} \frac{r}{R} \right) \int_{-R}^r d\zeta e^{-\zeta^2/(4t)} \int_{\{|\hat{\zeta}| > ((r-\zeta)\frac{R}{2})^{1/2}\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)} \\
& \geq |D - D_{R/2}| - (4\pi t)^{-1/2} \mathcal{H}^{m-1}(\partial D) \int_0^\infty dr \int_r^\infty d\zeta e^{-\zeta^2/(4t)} \\
& \quad - (4\pi t)^{-1/2} (m-1)2^{m-1} \mathcal{H}^{m-1}(\partial D) R^{-1} \int_0^\infty dr r \int_r^\infty d\zeta e^{-\zeta^2/(4t)} \\
& \quad - \frac{\sqrt{2}}{2} |D - D_{R/2}| e^{-R^2/(8t)} - (4\pi t)^{-m/2} (1 + (m-1)2^{m-2}) \mathcal{H}^{m-1}(\partial D) \\
& \quad \times \int_{-\infty}^\infty dr \int_0^\infty d\theta e^{-(r-\theta)^2/(4t)} \int_{\{|\hat{\zeta}| > (R\theta/2)^{1/2}\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)} \\
& = |D - D_{R/2}| - \pi^{-1/2} \mathcal{H}^{m-1}(\partial D) t^{1/2} - (m-1)2^{m-2} \mathcal{H}^{m-1}(\partial D) R^{-1} t \\
& \quad - \frac{\sqrt{2}}{2} |D - D_{R/2}| e^{-R^2/(8t)} \\
& \quad - (4\pi t)^{-(m-1)/2} (1 + (m-1)2^{m-2}) \mathcal{H}^{m-1}(\partial D) \int_0^\infty d\theta \int_{\{|\hat{\zeta}| > (R\theta/2)^{1/2}\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)} \\
& = |D - D_{R/2}| - \pi^{-1/2} \mathcal{H}^{m-1}(\partial D) t^{1/2} - (m-1)2^{m-2} \mathcal{H}^{m-1}(\partial D) R^{-1} t \\
& \quad - \frac{\sqrt{2}}{2} |D - D_{R/2}| e^{-R^2/(8t)} - 4(m-1)(1 + (m-1)2^{m-2}) \mathcal{H}^{m-1}(\partial D) R^{-1} t.
\end{aligned}$$

If  $D$  is not connected, then  $D = \cup_{i=1}^N D^i$  for some  $N \in \mathbb{N}$  and the above result holds for each component  $D^i$  of  $D$ . By taking the sum of the above result for each component  $D^i$

over  $i = 1, \dots, N$ , we obtain

$$\begin{aligned}
& \int_D dx \int_D dy p(x, y; t) \\
& \geq \sum_{i=1}^N \left( |D_{R/2}^i| - 2^{m/2} |D_{R/2}^i| e^{-R^2/(32t)} + \int_{D^i - D_{R/2}^i} dx \int_{B_1^i} dy p(x, y; t) \right) \\
& = |D_{R/2}| - 2^{m/2} |D_{R/2}| e^{-R^2/(32t)} + \sum_{i=1}^N \int_{D^i - D_{R/2}^i} dx \int_{B_1^i} dy p(x, y; t) \\
& \geq |D| - \pi^{-1/2} \mathcal{H}^{m-1}(\partial D) t^{1/2} - 2^{m/2} |D_{R/2}| e^{-R^2/(32t)} \\
& \quad - (m-1) 2^{m-2} \mathcal{H}^{m-1}(\partial D) R^{-1} t - \frac{\sqrt{2}}{2} |D - D_{R/2}| e^{-R^2/(8t)} \\
& \quad - 4(m-1)(1 + (m-1) 2^{m-2}) \mathcal{H}^{m-1}(\partial D) R^{-1} t.
\end{aligned}$$

□

**Lemma 13.** *Let  $D \subset \mathbb{R}^m$ ,  $m \geq 2$ , be an open set with  $R$ -smooth boundary  $\partial D$  and finite Lebesgue measure  $|D|$ . Then*

$$\begin{aligned}
H_D(t) & \leq |D| - \pi^{-1/2} \mathcal{H}^{m-1}(\partial D) t^{1/2} + 2\pi^{-1/2} \mathcal{H}^{m-1}(\partial D) t^{1/2} e^{-R^2/(32t)} \\
& \quad + 2^{-1}(m-1) \mathcal{H}^{m-1}(\partial D) R^{-1} t + \frac{\sqrt{2}}{2} |D - D_{R/2}| e^{-R^2/(8t)} \\
& \quad + (m-1)(1 + (m-1) 2^{m-2}) \mathcal{H}^{m-1}(\partial D) R^{-1} t.
\end{aligned}$$

*Proof.* First suppose that  $D$  is connected. If  $\delta(x) > \frac{R}{2}$  then

$$\int_{D_{R/2}} dx \int_D dy p(x, y; t) \leq \int_{D_{R/2}} dx \int_{\mathbb{R}^m} dy p(x, y; t) = |D_{R/2}|.$$

If  $\delta(x) < \frac{R}{2}$ , integrating the result of Lemma 11 with respect to  $x$  over  $D - D_{R/2}$  and using Lemma 6.7 in [6] we obtain that

$$\begin{aligned}
& \int_{D - D_{R/2}} dx \int_D dy p(x, y; t) \\
& \leq |D - D_{R/2}| - (4\pi t)^{-1/2} \int_{D - D_{R/2}} dx \int_{\delta(x)}^\infty d\zeta e^{-\zeta^2/(4t)} + \frac{\sqrt{2}}{2} |D - D_{R/2}| e^{-R^2/(8t)} \\
& \quad + (4\pi t)^{-m/2} \int_{D - D_{R/2}} dx \int_{\delta(x)}^{\delta(x)+R} d\zeta e^{-\zeta^2/(4t)} \int_{\{|\hat{\zeta}| > ((\zeta - \delta(x))R)^{1/2}\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)} \\
& = |D - D_{R/2}| - (4\pi t)^{-1/2} \int_0^{R/2} dr \mathcal{H}^{m-1}(\partial D_r) \int_r^\infty d\zeta e^{-\zeta^2/(4t)} \\
& \quad + \frac{\sqrt{2}}{2} |D - D_{R/2}| e^{-R^2/(8t)} \\
& \quad + (4\pi t)^{-m/2} \int_0^{R/2} dr \mathcal{H}^{m-1}(\partial D_r) \int_r^{r+R} d\zeta e^{-\zeta^2/(4t)} \int_{\{|\hat{\zeta}| > ((\zeta - r)R)^{1/2}\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)}
\end{aligned}$$

$$\begin{aligned}
&\leq |D - D_{R/2}| - (4\pi t)^{-1/2} \mathcal{H}^{m-1}(\partial D) \int_0^{R/2} dr \left( \frac{R-r}{R} \right)^{m-1} \int_r^\infty d\zeta e^{-\zeta^2/(4t)} \\
&\quad + \frac{\sqrt{2}}{2} |D - D_{R/2}| e^{-R^2/(8t)} + (4\pi t)^{-m/2} \mathcal{H}^{m-1}(\partial D) \int_0^{R/2} dr \left( \frac{R}{R-r} \right)^{m-1} \\
&\quad \times \int_r^{r+R} d\zeta e^{-\zeta^2/(4t)} \int_{\{|\hat{\zeta}| > ((\zeta-r)R)^{1/2}\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)} \\
&\leq |D - D_{R/2}| - (4\pi t)^{-1/2} \mathcal{H}^{m-1}(\partial D) \int_0^{R/2} dr \left( 1 - (m-1) \frac{r}{R} \right) \int_r^\infty d\zeta e^{-\zeta^2/(4t)} \\
&\quad + \frac{\sqrt{2}}{2} |D - D_{R/2}| e^{-R^2/(8t)} + (4\pi t)^{-m/2} \mathcal{H}^{m-1}(\partial D) \\
&\quad \times \int_0^{R/2} dr \left( 1 + (m-1) 2^{m-1} \frac{r}{R} \right) \int_r^{r+R} d\zeta e^{-\zeta^2/(4t)} \int_{\{|\hat{\zeta}| > ((\zeta-r)R)^{1/2}\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)} \\
&\leq |D - D_{R/2}| - (4\pi t)^{-1/2} \mathcal{H}^{m-1}(\partial D) \int_0^\infty dr \int_r^\infty d\zeta e^{-\zeta^2/(4t)} \\
&\quad + (4\pi t)^{-1/2} \mathcal{H}^{m-1}(\partial D) \int_{R/2}^\infty dr \int_r^\infty d\zeta e^{-\zeta^2/(4t)} \\
&\quad + (4\pi t)^{-1/2} (m-1) \mathcal{H}^{m-1}(\partial D) R^{-1} \int_0^\infty dr r \int_r^\infty d\zeta e^{-\zeta^2/(4t)} \\
&\quad + \frac{\sqrt{2}}{2} |D - D_{R/2}| e^{-R^2/(8t)} + (4\pi t)^{-m/2} \mathcal{H}^{m-1}(\partial D) (1 + (m-1) 2^{m-2}) \\
&\quad \times \int_0^\infty dr \int_0^R d\theta e^{-(r+\theta)^2/(4t)} \int_{\{|\hat{\zeta}| > (\theta R)^{1/2}\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)} \\
&\leq |D - D_{R/2}| - \pi^{-1/2} \mathcal{H}^{m-1}(\partial D) t^{1/2} + 2\pi^{-1/2} \mathcal{H}^{m-1}(\partial D) t^{1/2} e^{-R^2/(32t)} \\
&\quad + 2^{-1} (m-1) \mathcal{H}^{m-1}(\partial D) R^{-1} t + \frac{\sqrt{2}}{2} |D - D_{R/2}| e^{-R^2/(8t)} \\
&\quad + 2^{-1} (1 + (m-1) 2^{m-2}) (4\pi t)^{-(m-1)/2} \mathcal{H}^{m-1}(\partial D) \\
&\quad \times \int_0^\infty d\theta \int_{\{|\hat{\zeta}| > (\theta R)^{1/2}\}} d\hat{\zeta} e^{-|\hat{\zeta}|^2/(4t)} \\
&= |D - D_{R/2}| - \pi^{-1/2} \mathcal{H}^{m-1}(\partial D) t^{1/2} + 2\pi^{-1/2} \mathcal{H}^{m-1}(\partial D) t^{1/2} e^{-R^2/(32t)} \\
&\quad + 2^{-1} (m-1) \mathcal{H}^{m-1}(\partial D) R^{-1} t + \frac{\sqrt{2}}{2} |D - D_{R/2}| e^{-R^2/(8t)} \\
&\quad + (m-1) (1 + (m-1) 2^{m-2}) \mathcal{H}^{m-1}(\partial D) R^{-1} t.
\end{aligned}$$

If  $D$  is not connected, then  $D = \cup_{i=1}^N D^i$  for some  $N \in \mathbb{N}$  and the above result holds for each component  $D^i$  of  $D$ . By taking the sum of the above result for each component  $D^i$  over  $i = 1, \dots, N$ , we obtain;

$$\begin{aligned}
&\int_D dx \int_D dy p(x, y; t) \leq \sum_{i=1}^N \left( |D_{R/2}^i| + \int_{D^i - D_{R/2}^i} dx \int_{\mathbb{R}^m - B_2^i} dy p(x, y; t) \right) \\
&= |D_{R/2}| + \sum_{i=1}^N \int_{D^i - D_{R/2}^i} dx \int_{\mathbb{R}^m - B_2^i} dy p(x, y; t) \\
&\leq |D| - \pi^{-1/2} \mathcal{H}^{m-1}(\partial D) t^{1/2} + 2\pi^{-1/2} \mathcal{H}^{m-1}(\partial D) t^{1/2} e^{-R^2/(32t)} \\
&\quad + 2^{-1} (m-1) \mathcal{H}^{m-1}(\partial D) R^{-1} t + \frac{\sqrt{2}}{2} |D - D_{R/2}| e^{-R^2/(8t)} \\
&\quad + (m-1) (1 + (m-1) 2^{m-2}) \mathcal{H}^{m-1}(\partial D) R^{-1} t.
\end{aligned}$$

□

Now using Lemma 12, Lemma 13 and (22), we obtain that

$$\left| H_D(t) - |D| + \pi^{-1/2} \mathcal{H}^{m-1}(\partial D) t^{1/2} \right| \leq m^3 2^{m+2} |D| R^{-2} t,$$

which completes the proof of Theorem 2.

#### 4. PROOFS OF THEOREM 4, PROPOSITION 5 AND THEOREM 7.

To prove part (i) of Theorem 4 we suppose the integrability condition in the hypothesis of Theorem 4 holds. We let  $R > 0$  and  $\alpha \in (0, 1)$ . For  $n \in \mathbb{N}$ , we let

$$D_n = D \cap B(0; n).$$

Then

$$\begin{aligned} H_{D_n}(t) &= \int_{D_n} dx \int_{D_n} dy p(x, y; t) \\ &= \int_{D_n} dx \int_{D_n \cap B(x; R)} dy p(x, y; t) + \int_{D_n} dx \int_{D_n \cap (\mathbb{R}^m - B(x; R))} dy p(x, y; t) \\ &\leq (4\pi t)^{-m/2} \int_{D_n} dx \mu_{D_n}(x; R) \\ &\quad + (1 - \alpha)^{-m/2} \int_{D_n} dx e^{-\alpha R^2/(4t)} \int_{D_n} dy p(x, y; (1 - \alpha)^{-1} t) \\ &= (4\pi t)^{-m/2} \int_{D_n} dx \mu_{D_n}(x; R) + (1 - \alpha)^{-m/2} e^{-\alpha R^2/(4t)} H_{D_n}((1 - \alpha)^{-1} t) \\ &\leq (4\pi t)^{-m/2} \int_{D_n} dx \mu_{D_n}(x; R) + (1 - \alpha)^{-m/2} e^{-\alpha R^2/(4t)} H_{D_n}(t), \end{aligned}$$

where we have used that  $t \mapsto H_{D_n}(t)$  is decreasing. We now choose  $\alpha$  and  $R$  such that  $(1 - \alpha)^{-m/2} e^{-\alpha R^2/(4t)} < 1$ . This is clearly satisfied for  $R = (8mt)^{1/2}$  and  $\alpha = \frac{3}{4}$ . Since  $H_{D_n}(t) \leq |B(0; n)| < \infty$ , we may rearrange the terms and obtain that

$$\begin{aligned} H_{D_n}(t) &\leq (4\pi t)^{-m/2} \left( 1 - 2^m e^{-3m/2} \right)^{-1} \int_{D_n} dx \mu_{D_n}(x; (8mt)^{1/2}) \\ &= c_2(m) t^{-m/2} \int_{D_n} dx \mu_{D_n}(x; (8mt)^{1/2}) \\ (23) \quad &\leq c_2(m) t^{-m/2} \int_D dx \mu_D(x; (8mt)^{1/2}). \end{aligned}$$

For  $(x, y) \in \mathbb{R}^{2m}$ , we let

$$f_n(x, y) = p(x, y; t) \mathbf{1}_{D_n}(x) \mathbf{1}_{D_n}(y).$$

Then  $(f_n)$  is a monotone increasing sequence of non-negative functions, converging pointwise to  $p(x, y; t) \mathbf{1}_D(x) \mathbf{1}_D(y)$ . The Monotone Convergence Theorem applied to  $(f_n)$  with product measure  $dx dy$  gives that  $\lim_{n \rightarrow \infty} H_{D_n}(t) = H_D(t)$ . This together with (23) implies the right-hand side of (9).

To prove the lower bound in Theorem 4 we have for  $R > 0$ ,

$$\begin{aligned} H_D(t) &\geq \int_D dx \int_{D \cap B(x; R)} dy p(x, y; t) \\ &\geq (4\pi t)^{-m/2} e^{-R^2/(4t)} \int_D dx \int_{D \cap B(x; R)} dy \\ &= (4\pi t)^{-m/2} e^{-R^2/(4t)} \int_D dx \mu_D(x; R). \end{aligned}$$

The choice  $R = (8mt)^{1/2}$  gives the lower bound in (9).

To prove part (ii) we let  $\alpha = \frac{t_2}{t_1} \leq 1$ , and suppose  $H_D(t_1) < \infty$ . Then following (23), we have that

$$\begin{aligned} H_{D_n}(\alpha t_1) &\leq c_2(m)(\alpha t_1)^{-m/2} \int_D dx \mu_D(x; (8m\alpha t_1)^{1/2}) \\ &\leq c_2(m)t_2^{-m/2} \int_D dx \mu_D(x; (8mt_1)^{1/2}). \end{aligned}$$

Letting  $n \rightarrow \infty$  we obtain that

$$(24) \quad H_D(t_2) \leq c_2(m)t_2^{-m/2} \int_D dx \mu_D(x; (8mt_1)^{1/2}).$$

By the first inequality in (9) we also have that

$$(25) \quad \int_D dx \mu_D(x; (8mt_1)^{1/2}) \leq c_1(m)^{-1} t_1^{m/2} H_D(t_1),$$

and (10) follows from (24) and (25).

*Proof of Proposition 5.* We bound  $\int_{\Omega(\alpha, \Sigma)} dx dx' \mu_{\Omega(\alpha, \Sigma)}((x, x'); (8mt)^{1/2})$  from below, and follow the notation of (13). We restrict the integral such that  $\text{diam}(x^{-\alpha}\Sigma) \leq (4mt)^{1/2}$ . That is

$$(26) \quad x \geq \left( \frac{\text{diam}(\Sigma)}{(4mt)^{1/2}} \right)^{1/\alpha} := x_0(t).$$

We now choose  $t$  such that the above set of  $x$  satisfies  $x \geq 1$ . That is

$$t \leq \frac{1}{4m} \text{diam}(\Sigma)^2.$$

We will choose  $c > 0$  such that for any  $x' \in x^{-\alpha}\Sigma$  the cylinder

$$(x, x+c) \times (x+c)^{-\alpha}\Sigma \subset B((x, x'); (8mt)^{1/2}) \cap \Omega(\alpha, \Sigma).$$

Suppose  $(z, z') \in (x, x+c) \times (x+c)^{-\alpha}\Sigma$ . Then  $z' \in x^{-\alpha}\Sigma$  and  $|(z, z') - (x, x')|^2 = (|z-x|^2 + |z'-x'|^2) \leq c^2 + (\text{diam}(x^{-\alpha}\Sigma))^2 \leq c^2 + 4mt$ . We conclude that  $(z, z') \in B((x, x'); (8mt)^{1/2})$  for  $c = (4mt)^{1/2}$ . Since  $\Omega(\alpha, \Sigma)$  is horn-shaped,  $(x, x+c) \times (x+c)^{-\alpha}\Sigma \subset \Omega(\alpha, \Sigma)$ . We have that for all  $x$  satisfying (26)

$$\begin{aligned} \mu_{\Omega(\alpha, \Sigma)}((x, x'); (8mt)^{1/2}) &\geq (4mt)^{1/2} \mathcal{H}^{m-1}((x + (4mt)^{1/2})^{-\alpha}\Sigma) \\ &= (4mt)^{1/2} (x + (4mt)^{1/2})^{-(m-1)\alpha} \mathcal{H}^{m-1}(\Sigma). \end{aligned}$$

We have that

$$\begin{aligned} &\int_{\Omega(\alpha, \Sigma)} dx dx' \mu_{\Omega(\alpha, \Sigma)}((x, x'); (8mt)^{1/2}) \\ &\geq \int_{(x_0(t), \infty)} dx \int_{x^{-\alpha}\Sigma} dx' (4mt)^{1/2} (x + (4mt)^{1/2})^{-(m-1)\alpha} \mathcal{H}^{m-1}(\Sigma) \\ &= \int_{(x_0(t), \infty)} dx (4mt)^{1/2} x^{-(m-1)\alpha} (x + (4mt)^{1/2})^{-(m-1)\alpha} \mathcal{H}^{m-1}(\Sigma)^2. \end{aligned}$$

The integral diverges for  $0 < \alpha \leq (2(m-1))^{-1}$ . For  $(2(m-1))^{-1} < \alpha < (m-1)^{-1}$  we see that the right-hand side above is of order  $x_0(t)^{1-2(m-1)\alpha}$ . Using (26) and the lower bound in Theorem 4 we conclude that there exists  $C_1(\alpha, m, \Sigma) > 0$  such that for all  $t$  sufficiently small

$$(27) \quad H_{\Omega(\alpha, \Sigma)}(t) \geq C_1(\alpha, m, \Sigma) t^{((m-1)\alpha-1)/(2\alpha)}.$$

Next we obtain an upper bound for  $\int_{\Omega(\alpha, \Sigma)} dx dx' \mu_{\Omega(\alpha, \Sigma)}((x, x'); (8mt)^{1/2})$ . We note that for all  $x' \in x^{-\alpha}\Sigma$  and  $x \geq 1 + (8mt)^{1/2}$ ,

$$(x - (8mt)^{1/2}, x + (8mt)^{1/2}) \times (x - (8mt)^{1/2})^{-\alpha}\Sigma \supset B((x, x'); (8mt)^{1/2}) \cap \Omega(\alpha, \Sigma).$$

Hence

$$\begin{aligned} \mu_{\Omega(\alpha, \Sigma)}((x, x'); (8mt)^{1/2}) &\leq 2(8mt)^{1/2} \mathcal{H}^{m-1}((x - (8mt)^{1/2})^{-\alpha}\Sigma) \\ &= 2(8mt)^{1/2} (x - (8mt)^{1/2})^{-(m-1)\alpha} \mathcal{H}^{m-1}(\Sigma). \end{aligned}$$

We let  $x_0(t)$  be as in (26),  $(2(m-1))^{-1} < \alpha < (m-1)^{-1}$ , and let  $t > 0$  be such that  $x_0(t) \geq 1 + (8mt)^{1/2}$ . Then

$$\begin{aligned}
& \int_{\{(x,x') \in \Omega(\alpha, \Sigma) : x > x_0(t)\}} dx dx' \mu_{\Omega(\alpha, \Sigma)}((x, x'); (8mt)^{1/2}) \\
& \leq 2(8mt)^{1/2} \int_{x_0(t)}^{\infty} dx \int_{x-\alpha\Sigma}^{\infty} dx' (x - (8mt)^{1/2})^{-(m-1)\alpha} \mathcal{H}^{m-1}(\Sigma) \\
& = 2(8mt)^{1/2} \int_{x_0(t)}^{\infty} dx x^{-(m-1)\alpha} (x - (8mt)^{1/2})^{-(m-1)\alpha} \mathcal{H}^{m-1}(\Sigma)^2 \\
& \leq 2(8mt)^{1/2} (2(m-1)\alpha - 1)^{-1} (x_0(t) - (8mt)^{1/2})^{1-2(m-1)\alpha} \mathcal{H}^{m-1}(\Sigma)^2 \\
(28) \quad & \leq C_2(\alpha, m, \Sigma) t^{((2m-1)\alpha-1)/(2\alpha)}.
\end{aligned}$$

for some finite constant  $C_2(\alpha, m, \Sigma)$  and all  $t$  sufficiently small.

For all  $(x, x') \in \Omega(\alpha, \Sigma)$  with  $x \leq x_0(t)$  we bound

$$\mu_{\Omega(\alpha, \Sigma)}((x, x'); (8mt)^{1/2}) \leq \omega_m (8mt)^{m/2}.$$

Hence

$$\begin{aligned}
& \int_{\{(x,x') \in \Omega(\alpha, \Sigma) : 1 < x < x_0(t)\}} dx dx' \mu_{\Omega(\alpha, \Sigma)}((x, x'); (8mt)^{1/2}) \\
& \leq \omega_m (8mt)^{m/2} |\{\Omega(\alpha, \Sigma) : 1 < x < x_0(t)\}| \\
& = \omega_m (8mt)^{m/2} (1 - (m-1)\alpha)^{-1} \left( x_0(t)^{1-(m-1)\alpha} - 1 \right) \mathcal{H}^{m-1}(\Sigma) \\
& \leq \omega_m (8mt)^{m/2} (1 - (m-1)\alpha)^{-1} x_0(t)^{1-(m-1)\alpha} \mathcal{H}^{m-1}(\Sigma) \\
(29) \quad & = C_3(\alpha, m, \Sigma) t^{((2m-1)\alpha-1)/(2\alpha)}.
\end{aligned}$$

By Theorem 4, (28) and (29) we conclude that

$$(30) \quad H_{\Omega(\alpha, \Sigma)}(t) \leq c_2(m) (C_2(\alpha, m, \Sigma) + C_3(\alpha, m, \Sigma)) t^{((m-1)\alpha-1)/(2\alpha)}.$$

Proposition 5 now follows by (27) and (30).  $\square$

The following is a crucial ingredient in the proof of Theorem 7.

**Proposition 14.** *Let  $D$  be an open set in  $\mathbb{R}^m$  with finite Lebesgue measure, and let  $s \geq 0$ ,  $t \geq 0$ . Then*

$$(31) \quad F_D(s+t) \leq F_D(s) + F_D(t).$$

*Proof.* By the definition of  $F_D(t)$  in (16), and (5) we recover Preunkert's formula (see [15]):

$$\begin{aligned}
F_D(t) &= \int_{\mathbb{R}^m} dx \int_D dy p(x, y; t) - \int_D dx \int_D dy p(x, y; t) \\
(32) \quad &= \int_{\mathbb{R}^m - D} dx \int_D dy p(x, y; t).
\end{aligned}$$

By the heat semigroup property we have that

$$(33) \quad p(x, y; s+t) = \int_{\mathbb{R}^m} dz p(x, z; s) p(z, y; t).$$

By (32), (33) and Tonelli's Theorem we have that

$$\begin{aligned}
F_D(s+t) &= \int_{\mathbb{R}^m-D} dx \int_D dy p(x,y;s+t) \\
&= \int_{\mathbb{R}^m} dz \int_{\mathbb{R}^m-D} dx \int_D dy p(x,z;s)p(z,y;t) \\
&= \int_D dz \int_{\mathbb{R}^m-D} dx \int_D dy p(x,z;s)p(z,y;t) \\
&\quad + \int_{\mathbb{R}^m-D} dz \int_{\mathbb{R}^m-D} dx \int_D dy p(x,z;s)p(z,y;t) \\
&= \int_D dz \int_{\mathbb{R}^m-D} dx p(x,z;s)u_D(z;t) \\
&\quad + \int_{\mathbb{R}^m-D} dz \int_D dy p(z,y;t)u_{\mathbb{R}^m-D}(z;s) \\
&\leq \int_D dz \int_{\mathbb{R}^m-D} dx p(x,z;s) + \int_{\mathbb{R}^m-D} dz \int_D dy p(z,y;t) \\
&= F_D(s) + F_D(t).
\end{aligned}$$

□

We note that in the proof above we did not use the Euclidean structure. A similar statement and proof would hold for open sets  $D$  in complete Riemannian manifolds which are stochastically complete. See Section 3.3 in [14] for details on stochastic completeness.

*Proof of Theorem 7.* To prove the lower bound we have by (32) that for any  $R > 0$ ,

$$\begin{aligned}
F_D(t) &\geq \int_D dx \int_{(\mathbb{R}^m-D) \cap B(x;R)} dy p(x,y;t) \\
&\geq (4\pi t)^{-m/2} e^{-R^2/(4t)} \int_D dx \nu_D(x;R).
\end{aligned}$$

The choice  $R = 4(mt)^{1/2}$  yields the lower bound in (17).

To prove the upper bound we have the following estimate.

$$\begin{aligned}
F_D(t) &= \int_D dx \int_{(\mathbb{R}^m-D) \cap B(x;R)} dy p(x,y;t) \\
&\quad + \int_D dx \int_{(\mathbb{R}^m-D) \cap (\mathbb{R}^m-B(x;R))} dy p(x,y;t) \\
&\leq (4\pi t)^{-m/2} \int_D dx \nu_D(x;R) \\
&\quad + 2^{m/2} e^{-R^2/(8t)} \int_D dx \int_{(\mathbb{R}^m-D) \cap (\mathbb{R}^m-B(x;R))} dy p(x,y;2t) \\
&\leq (4\pi t)^{-m/2} \int_D dx \nu_D(x;R) + 2^{m/2} e^{-R^2/(8t)} F_D(2t).
\end{aligned}$$

From (31) we infer that  $F_D(2t) \leq 2F_D(t)$ . We conclude that

$$(34) \quad F_D(t) \leq (4\pi t)^{-m/2} \int_D dx \nu_D(x;R) + 2^{(2+m)/2} e^{-R^2/(8t)} F_D(t).$$

We now choose  $R = 4(mt)^{1/2}$  so that  $2^{(2+m)/2} e^{-R^2/(8t)} < 1$ . Rearranging terms in (34) yields the upper bound in (17). □

## REFERENCES

- [1] Aikawa, H., Kilpeläinen, T., Shanmugalingam, N., Zhong, X.: Boundary Harnack principle for  $p$ -harmonic Functions in smooth Euclidean domains. *Potential Anal.* **26** (2007) 281–301.
- [2] van den Berg, M.: On the Asymptotics of the heat equation and bounds on traces associated with the Dirichlet Laplacian. *J. Funct. Anal.* **71** (1987) 279–293.
- [3] van den Berg, M.: Heat content and Hardy inequality for complete Riemannian manifolds. *J. Funct. Anal.* **233** (2006) 478–493.
- [4] van den Berg, M.: Heat flow and Hardy inequality in complete Riemannian manifolds with singular initial conditions. *J. Funct. Anal.* **250** (2007) 114–131.

- [5] van den Berg, M.: Heat flow and perimeter in  $\mathbb{R}^m$ . *Potential Anal.* **39** (2013) 369–387.
- [6] van den Berg, M., Davies, E.B.: Heat flow out of regions in  $\mathbb{R}^m$ . *Mathematische Zeitschrift* **202** (1989) 463–482.
- [7] van den Berg, M., Gilkey, P. B.: Heat content and a Hardy inequality for complete Riemannian manifolds. *Bull. London Math. Soc.* **36** (2004) 577–586.
- [8] van den Berg, M., Gilkey, P.: Heat flow out of a compact manifold. *J. Geometric Analysis*, DOI 10.1007/s12220-014-9485-2.
- [9] van den Berg, M., Gilkey, P.: Heat content with singular initial temperature and singular specific heat. *Potential Anal.* **42** (2015) 1–38.
- [10] van den Berg, M., Gilkey, P., Grigor'yan, A., Kirsten, K.: Hardy inequality and heat semigroup estimates for Riemannian manifolds with singular data. *Comm. Partial Differential Equations* **37** (2012) 885–900.
- [11] van den Berg, M., Gilkey, P., Kang, H.: Neumann heat content asymptotics with singular initial temperature and singular specific heat. *Journal of Fixed Point Theory and Applications* **14** (2013) 267–298.
- [12] Evans, L. C.: *Partial Differential Equations*. Graduate Studies in Mathematics, **19**, American Mathematical Society, Providence RI, (2002).
- [13] Evans, L. C., Gariepy, R.F.: *Measure Theory and Fine Properties of Functions*. Chapman and Hall/ CRC, Boca Raton (1992).
- [14] Grigor'yan, A.: Estimates of heat kernels on Riemannian manifolds, in *Spectral Theory and Geometry*. ICMS Instructional Conference, Edinburgh, 1998. *London Math. Soc. Lecture Note Ser.*, **273**, Cambridge Univ. Press, Cambridge, (1999) 140–225.
- [15] Preunkert, M.: A semigroup version of the isoperimetric inequality. *Semigroup Forum* **68** (2004) 233–245.
- [16] Miranda, M. Jr., Pallara, D., Paronetto, F., Preunkert, M.: Short-time heat flow and functions of bounded variation in  $\mathbb{R}^N$ . *Annales de la Faculté des Sciences de Toulouse* **16** (2007) 125–145.

SCHOOL OF MATHEMATICS, UNIVERSITY OF BRISTOL, UNIVERSITY WALK, BRISTOL, BS8 1TW, U.K.